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An Oscillation Criterion for Second Order Nonlinear Differential Equations with Functional Arguments*

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Best possible conditions are given here, under which all solutions of the equation $y''(t) + p(t)f(y(t), y(g(t))) = 0$ are oscillatory.

Consider the linear differential equation

$$y''(t) + a(t)\,y(t) = 0, \tag{1}$$

where $a(t) \in C[t_0, \infty)$. By the well-known theorem of Winter [6],

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} du \int_{t_0}^{t} a(s)\,ds = \infty \tag{2}$$

is sufficient for Eq. (1) to be oscillatory even when $a(t)$ is not assumed positive. Hartman [2] has shown that the limit can not be replaced by the upper limit in condition (2).

Considering the equation

$$y''(t) + \lambda b(t)\,y(t) = 0, \tag{3}$$

where $b(t) \in C[t_0, \infty)$, we call $b(t)$ a strongly oscillatory coefficient if (3) is oscillatory for all positive λ. If $b(t) \geqslant 0$, Nehari [4] shows that

$$\lim_{t\to\infty} t \int_{t}^{\infty} b(s)\,ds = \infty \tag{4}$$

is a necessary and sufficient condition for $b(t)$ to be a strongly oscillatory coefficient.

We consider the equation

$$y''(t) + p(t)\,f(y(t), y(g(t))) = 0, \tag{5}$$

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where p, $g \in C[t_0, \infty)$, $f \in C(R \times R)$, $R = (-\infty, \infty)$ and $f(y_1, y_2)$ has the sign of y_1 and y_2 when they have the same sign.

Travis [5] has recently demonstrated that all solutions of (5) are oscillatory if

(C_1) $h(t) \leqslant g(t)$ and $0 < k \leqslant h'(t) \leqslant 1$,

(C_2) there exists $M > 0$ such that $y \geqslant M$ implies

$$\liminf_{|w| \to \infty} \left| \frac{f(y, w)}{w} \right| \geqslant c > 0,$$

(C_3) $p(t) \geqslant 0$ and $\limsup_{t \to \infty} t \int_t^\infty p(s)\, ds = \infty$.

Our main purpose in this paper is to give a new integral criterion for the oscillation of Eq. (5), based on the use of the nth primitive

$$A_n(t) = \frac{1}{(n!)} \int_{t_0}^{t} (t - u)^{n-1} p(u)\, du$$

of the coefficient $p(t)$, which has the critetia of Winter [6] and Travis [5] as a particular case.

THEOREM 1. *Let* (C_1) *and* (C_2) *hold. If* $p(t) \geqslant 0$ *and*

$$\limsup_{t \to \infty} t^{1-n} A_n(t) = \infty, \tag{C}$$

where $A_n(t)$ *is the* nth *primitive of* $p(t)$ *for some* $n > 2$, *then all solutions of* (5) *are oscillatory.*

Proof. Assume that $y(t)$ is a nonoscillatory solution of (5). Without loss of generality, we can assume that $y(t) > 0$ for large t. We see easily that $y'(t) > 0$ for large t. Let $w(t) = y'(t)/y(h(t))$. Then $w(t)$ satisfies

$$w'(t) = -p(t) \frac{f(y(t), y(g(t)))}{y(h(t))} - \frac{y'(h(t))}{y(h(t))} h'(t)\, w(t).$$

Since $y'(t) > 0$ for large t, $\lim_{t \to \infty} y(t)$ exists either as a finite or infinite limit. If $\lim_{t \to \infty} y(t) = b$ is finite, then

$$\lim_{t \to \infty} \frac{f(y(t), y(g(t)))}{y(g(t))} = \frac{f(b, b)}{b} > 0.$$

If $\lim_{t \to \infty} y(t) = \infty$, then by ($C_2$) we have that

$$\frac{f(y(t), y(g(t)))}{y(g(t))} \geqslant c > 0 \tag{6}$$

for large t. In either case, we have that (6) holds for t large enough. Since $y(t)$ is increasing for large t we have that

$$p(t)\frac{f(y(t), y(g(t)))}{y(h(t))} \geqslant p(t)\frac{f(y(t), y(g(t)))}{y(g(t))} \geqslant cp(t)$$

and

$$\frac{y'(h(t))}{y(h(t))} h'(t)\, w(t) \geqslant kw^2(t).$$

Thus

$$w'(t) \leqslant -cp(t) - kw^2(t),$$

whence it follows that

$$\int_{t_0}^{t} (t-u)^{n-1} w'(u)\, du \leqslant -c\int_{t_0}^{t} (t-u)^{n-1} p(u)\, du - k\int_{t_0}^{t} (t-u)^{n-1} w^2(u)\, du.$$

Since

$$\int_{t_0}^{t} (t-u)^{n-1} w'(u)\, du = (n-1)\int_{t_0}^{t} (t-u)^{n-2} w(u)\, du - w(t_0)(t-t_0)^{n-1},$$

we get

$$\begin{aligned}
ct^{1-n} & \int_{t_0}^{t} (t-u)^{n-1} p(u)\, du \\
& \leqslant w(t_0)\left(\frac{t-t_0}{t}\right)^{n-1} \\
& \quad - t^{1-n}\left[\int_{t_0}^{t} k(t-u)^{n-1} w^2(u)\, du + \int_{t_0}^{t} (n-1)(t-u)^{n-2} w(u)\, du\right] \\
& = w(t_0)\left(\frac{t-t_0}{t}\right)^{n-1} + 4^{-1}(n-1)^2 k^{-1}(n-2)^{-1} t^{1-n}(t-t_0)^{n-2} \\
& \quad - t^{1-n}\int_{t_0}^{t} [k^{1/2}w(u)(t-u)^{(1/2)(n-1)} + 2^{-1}k^{-1/2}(n-1)(t-u)^{(1/2)(n-3)}]^2\, du \\
& \leqslant w(t_0)\, t^{1-n}(t-t_0)^{n-1} + (4k)^{-1}(n-1)^2(n-2)^{-1} t^{1-n}(t-t_0)^{n-2} \to w(t_0)
\end{aligned}$$

as $t \to \infty$, which contradicts condition (C). Thus our proof is complete.

THEOREM 2. *Consider the equation*

$$y''(t) + p(t)\, F(y(t)) = 0, \tag{7}$$

where $p(t) \in C[t_0, \infty), F(y) \in C(R)$, $yF(y) > 0$ *for* $y \neq 0$, $F'(y)$ *exists and is continuous for* $y \in R' \equiv (-\infty, 0) \cup (0, \infty)$. *If* (C) *holds and*

$$F'(y) \geqslant k > 0 \qquad \text{for } y \in R',$$

then every solution of (7) *is oscillatory.*

Proof. Assume that $y(t)$ is a nonoscillatory solution of (7). Letting $w(t) = y'(t)/F(y(t))$, we have

$$w'(t) + w^2(t)\,F'(y(t)) + p(t) = 0.$$

Hence

$$\int_{t_0}^{t} (t-u)^{n-1}\,w'(u)\,du + \int_{t_0}^{t} (t-u)^{n-1}\,w^2(u)\,F'(y(u))\,du$$

$$= -\int_{t_0}^{t} (t-u)^{n-1}\,p(u)\,du.$$

As in the proof of Theorem 1, we have

$$t^{1-n} \int_{t_0}^{t} (t-u)^{n-1}\,p(u)\,du$$

$$\leqslant w(t_0)\,t^{1-n}(t-t_0)^{n-1} + (4k)^{-1}\,(n-2)^{-1}\,(n-1)^2\,t^{1-n}(t-t_0)^{n-2} \to w(t_0)$$

as $t \to \infty$, a contradiction. This contradiction completes our proof.

From Theorem 2, we have the following which is due to Kamenev [7].

COROLLARY. *If the condition* (C) *holds, then all solutions of the equation*

$$y''(t) + p(t)\,y(t) = 0$$

are oscillatory.

Remark 1. We see easily that if (2) holds, then (C) holds for $n = 3$. Thus the oscillation criterion of Winter [6] is a special case of our Theorem 2.

Remark 2. The domain of applicability of our oscillation criterion is wider than those for many criteria already known. For example, if $p(t)$ in Eq. (7) is such that $A_3(t) = e^{t^2} \sin t$, then our Theorem 2 says that Eq. (7) is oscillatory, whereas none of the known criteria [2, 3, 4, 6] can obtain this result.

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